MP8, MP3 (Grade 3)

Task: Adding 1 to an Addend, Adding 1 to a Factor

Practice standard focus: MP8: Mathematically proficient students at the elementary grades look for regularities as they solve multiple related problems, then identify and describe these regularities. . . . Mathematically proficient students formulate conjectures about what they notice . . . . As students practice articulating their observations, they learn to communicate with greater precision (MP6).
MP3: Construct viable arguments and critique the reasoning of others. Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. . . . Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems.

Content standard focus: 3.OA Represent and solve problems involving multiplication. 3.OA Understand properties of multiplication. 3.OA Identify and explain patterns in arithmetic.

Introduction

This extended example presents a sequence of eight lessons in which students 1) identify regularities they notice in pairs of related problems, 2) articulate a generalization about the behavior of an operation, 3) explore that generalization, and 4) develop arguments to prove that the generalization is true for all whole numbers. The first generalization they work on is: Given an addition expression, if 1 is added to an addend, the sum increases by 1. The second generalization is: Given a multiplication expression, if 1 is added to a factor, the sum increases by the other factor.

The example is conveyed through text and video. The text is drawn from the teacher's reports which were based on audio recordings[1]. Pseudonyms are used for the teacher and students that appear in the text. Actual names are used for students who appear in the video.

[1] Adapted and reprinted with permission from *Connecting Arithmetic to Algebra: Strategies for Building Algebraic Thinking in the Elementary Grades* by Susan Jo Russell, Deborah Schifter, and Virginia Bastable. Copyright © 2011 by TERC. Published by Heinemann, Portsmouth, NH. Pp. 117-123. Pseudonyms are used for teacher and students.

Sessions 1 and 2

The teacher, Alice Kaye, posted the following pairs of problems for the class to solve and asked them what they noticed about the problems.

7 + 5 = 12 7 + 6 = ___	7 + 5 = 12 8 + 5 = ___
9 + 4 = 13 9 + 5 = ___	9 + 4 = 13 10 + 4 = __

What do you notice?
What's happening here?

Ms. Kaye acknowledged that the numbers were not challenging to the class. She told them the purpose of the discussion was not to solve these problems, but to consider ideas that she wanted them to be able to say clearly and convincingly. The class filled in the blanks, and then began to talk about what they noticed. They mentioned that one number changes and one stays the same, and that the last number in the equation changed, too. The discussion went on like this for a few minutes, until one child, Evan, said, "Since 9 + 4 is 13, 9 + 5 has to be 1 more than 13."

Ms. Kaye realized that Evan's assertion, his expression of necessity, was significant, and she asked students to break into pairs to discuss what Evan meant. When they came back together, they continued to work on the ideas. Toward the end of the lesson, Pamela said, "I was just wondering. How did Evan come up with the idea he had? Because these are not just everyday ideas that you come up with every day."

Evan responded, "I'm not really sure. I just know it. It kind of seems obvious to me, so I didn't think to think about it before."

At the end of the discussion, the teacher gave the class directions to do one of two things: 1) to write down a statement if they have a way of putting the idea into words or 2) to come up with other pairs of equations that do the same thing. In this way, everyone in the class was working on the same idea, but she also made sure every student had a task that was accessible to him or her.

In preparation for the next class, Ms. Kaye went through the students' work and created two posters, one with more examples, the other with ways students tried to articulate the idea. In Session 2, students considered the different articulations of their evolving idea. At the end of this discussion, the class had come up with this statement: In addition, if you increase one of the addends by 1, then the sum will also increase by 1.

Commentary on Sessions 1 and 2

In these sessions, students were engaged in MP8: Look for and express regularity in repeated reasoning. Ms. Kaye had presented pairs of problems to draw their attention specifically to a generalization. After looking at patterns in the numbers, they realized there was a claim they could make about the behavior of addition. Then the challenge became not only to notice a pattern, but to figure out how to state the generalization. In this way, they were also engaged with MP6: Attend to precision. That is, the class worked to come up with a statement that clearly communicated the idea they were trying to express.

Note that the generalization the students were exploring can be viewed as a special case of the associative property of addition: $x + (y + 1) = (x + y) + 1$.

Session 3

Ms. Kaye gave her students the challenge to convince somebody else that their conjecture is true for all numbers. The class broke into pairs to work on this task. At the end of the session, they came back together to share and discuss their arguments.

Video: Adding 1 to an Addend, Adding 1 to a Factor, Part 1[2] (grade 3)
http://vimeo.com/66200697

The lesson continued for another 20 minutes in which other students shared their models and story contexts and discussed whether their arguments demonstrate the general claim in a way that would be convincing to another person who might enter the room.

Commentary on Session 3

In this session, students were engaged in MP3: *Construct viable arguments and critique the reasoning of others*. They had stated a conjecture—In addition, if you increase one of the addends by 1, then the sum will also increase by 1—and now they were working to prove that this is true for all whole numbers. The students in the video used cubes as concrete referents and acted on the cubes to demonstrate their conjecture. Using stacks of cubes to represent the two addends, they showed that when one cube is added to a stack (when you increase one addend by 1), in that same action the total number of cubes increases by 1 (the sum will also increase by 1). Although their representations necessarily showed a fixed number of cubes, the students explained that the number of cubes doesn't matter. No matter how many cubes are in the stacks, the same thing will happen.

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[2] Unpublished video from the project, *Using Routines as an Instructional Tool for Developing Elementary Students' Conceptions of Proof*.

The students in the class attended to one another's arguments. At the outset of the discussion, when Ms. Kaye asked what students notice about all the representations, Maddie pointed out that each representation has a 1 in it. After Lucas and Eli presented their argument, Oscar mentioned that his is similar to theirs.

Sessions 4 and 5

Ms. Kaye used the first three sessions to launch an exploration of multiplication: The lesson sequence continued as Ms. Kaye asked what happens when 1 is added to a factor. By contrasting what happens when 1 is added to an addend to what happens when 1 is added to a factor, she planned to highlight for her students differences between the two operations: What is true for addition isn't necessarily true for multiplication.

Ms. Kaye presented the class with these problems (including the products)

$7 \times 5 = 35$ $7 \times 6 = 42$	$7 \times 5 = 35$ $8 \times 5 = 40$
$9 \times 4 = 36$ $9 \times 5 = 45$	$9 \times 4 = 36$ $10 \times 4 = 40$

and provided the prompt, "In a multiplication problem, if you increase one of the factors by 1, I think this will happen to the product: ______________________."

The class found this task more challenging. Some students were, at first, surprised to discover that adding 1 to the product did not work, and some could not figure out what change was happening. Those who could identify the generalization articulated their conjectures. Here are several examples of how they articulated their ideas:

The number that is not increased is the number that the answer goes up by.
The number that is staying and not going up, increases by however many it is.
I think that the factor you increase, it goes up by the other factor.

In session 5, the class discussed the ideas from the previous day. As a group, they worked to understand the generalizations offered by individual students and tested out several more examples to see if their conjecture worked for other numbers.

Commentary on Sessions 4 and 5

Again, in these sessions, students were engaged with MP8: *Look for and express regularity in repeated reasoning*. At first, some of the students were surprised to realize multiplication behaves differently from addition and that they needed to come up with a new conjecture.

Note that the generalization is a special case of the distributive property of multiplication over addition:

$a(b + 1) = ab + a$
$(a + 1)b = ab + b$

Coming early in these students' learning about multiplication, the generalization was challenging to some members of the class. For this reason, even as some students could express the regularity to be seen in the problems, Ms. Kaye provided opportunity for students to continue to work with specific numbers.

Sessions 6, 7, and 8

For session 6, Ms. Kaye designed a task that would support students who were still sorting out what happens to the product when 1 is added to a factor and would also challenge those who were ready to work on a proof of their conjecture. She asked students to work in pairs on the following assignment.

> Choose which of the original equations you want to work with. Then do one of these…
>
> Draw a picture for the original equation; then change it just enough to match the new equations.
> Make an array for the original equation; then change it just enough to match the new equations.
> Write a story for the original equation; then change it just enough to match the new equations.
>
> Example: Original equation $7 \times 5 = 35$
>
> New equations $7 \times 6 = 42$
> $8 \times 5 = 40$

In session 7, the class came together to share their pictures and arrays and discuss what happens when 1 is added to a factor.

Video: Adding 1 to an Addend, Adding 1 to a Factor, Part 2[3]
http://vimeo.com/66217440

[3] Unpublished video from the project, *Using Routines as an Instructional Tool for Developing Elementary Students' Conceptions of Proof*.

Different members of the class presented their problems and models. Toward the end of the lesson, one student, Ayah, attempted to present an argument for the general claim.

Video: Adding 1 to an Addend, Adding 1 to a Factor, Part 3[4]
http://vimeo.com/66200698

The lesson concluded with one final question posed by Ms. Kaye.

Video: Adding 1 to an Addend, Adding 1 to a Factor, Part 4[5]
http://vimeo.com/66201471

In session 8, the class continued to discuss the stories, representations, and models created by pairs of students. Then they turned back to the question of whether they could use their models to make an argument in support of the claim that their rule will work for any numbers.

Commentary on Sessions 6, 7, and 8

In these sessions, students were engaged in MP3: *Construct viable arguments and critique the reasoning of others*. The conjecture about multiplication, which requires keeping track of multiple units, was harder for the class than the one about addition.

Most of the students' arguments applied to the specific problems. For example, Daniel and Riley showed why the product increased by 5 when the problem changed from 7 × 5 to 8 x 5—given 7 bowls of fish in the store, each bowl holding 5 fish, if one more bowl appears, increasing 7 to 8, the total number of fish increases by 5, the number of fish in own bowl—and why the product increased by 7 when the problem changed from 7 × 5 to 7 × 6—returning to 7 bowls of fish, each bowl holding 5 fish, if 1 fish is added to each bowl, increasing 6 to 7, the total number of fish increases by 7, the number of bowls.

Ayah attempted to present an argument for the general claim. She began, "Here we have some number of sticks and some number in each stick." However, as she moved into her explanation, she reverted back to the specific numbers in order to keep straight what was being added, how the addition changed one of the factors, and how it changed the product.

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[4] Unpublished video from the project, *Using Routines as an Instructional Tool for Developing Elementary Students' Conceptions of Proof*.
[5] Unpublished video from the project, *Using Routines as an Instructional Tool for Developing Elementary Students' Conceptions of Proof*.

Thus, all of the students in the class had an opportunity to represent and solve problems involving multiplication. They created contexts in which the total number of objects can be represented as 7×5, and represented the problems as equal groups and arrays.

Furthermore, all of the students in the class constructed arguments to show the special case: Given 7×5, they explained why the product must increase by 5 when 7 increases by 1 and why the product must increase by 7 when 5 increases by 1.

In addition, some of the students challenged themselves to prove their generalization: When a factor increases by 1, the product increases by the other factor. Although Ayah, in her attempt to prove the general claim, reverted back to using specific numbers, she made a significant step forward to understand what it takes to prove a general claim.

At the end of this sequence of lessons, students in the class could articulate what characterizes multiplication as distinct from addition. Abriana described what was different between the two explorations: “When we did ‘Something about Addition,’ there’s only 1 added on, and we didn’t have those groups…, but in [multiplication] we have groups, and you either add on a group or add 1 to each group.”